The Weierstrass function

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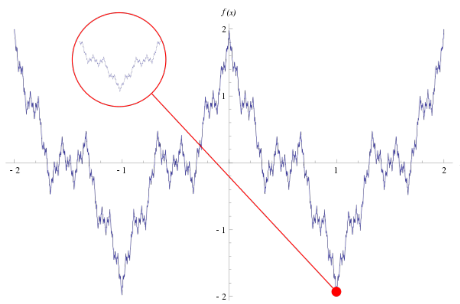

1 Introduction

The Weierstrass function is a real-valued function that is continuous everywhere, but differentiable nowhere. It is named after its discoverer, Karl Weierstrass, who published a paper on the subject to specifically challenge the notion that every continuous function is differentiable except on a set of isolated points.

Weierstrass presented his findings to the Prussian Academy of Sciences on July 1872. In his original paper, the Weierstrass function is defined as the Fourier series:

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \tag{1}$$

where $0 < a < 1$, b is a positive odd integer, and $ab > 1 + \frac{3}{2}\pi$

The Weierstrass function was one of the first fractals studied. The function has details at every level, so zooming in on a piece of the curve does not show it progressively getting closer and closer to a straight line. The discovery of a continuous, yet non-differentiable function upended mathematics at the time, as many proofs had assumed that continuous functions were always differentiable.

The proof below summarizes the proof of nowhere differentiability for the Weierstrass function, originally written by Jeff Calder, Associate Professor of Mathematics at the University of Minnesota.

2 Proof of the Weierstrass Function

2.1 Proving $W(x)$ is continuous everywhere

We first have to prove that $W(x)$ is continuous everywhere.

Since $|cos(x)| \leq 1$ for all $x \in \mathbb{R}$, we have

$$|a^n \cos(b^n \pi x)| = a^n |cos(b^n \pi x)| \leq a^n \tag{2}$$

Since the geometric series $\sum a^n$ converges for $a \in (0, 1)$, the Weierstrass M-test shows that $W(x)$ also converges uniformly. Since each function $a^n \cos(b^n \pi x)$ is continuous, **W is therefore continuous,** being the sum of continuous functions from 0 to n as n approaches infinity.

2.2 Establishing equations

Let $x, y \in \mathbb{R}$ and $x > y$. By the fundamental theorem of Calculus,

$$\cos(x) - \cos(y) = \int_y^x -\sin(t)dt \leq \int_y^x 1dt = x - y \tag{3}$$

and

$$\cos(x) - \cos(y) \geq \int_y^x -1dt = -(x - y) \tag{4}$$

which implies

$$|\cos(x) - \cos(y)| \leq |x - y| \tag{5}$$

Now consider $\cos(n\pi + x)$ for an integer n and $x \in \mathbb{R}$. By definition of natural numbers, n is either even or odd, which means $n = 2k$ or $n = 2k + 1$ for some natural number k.

Case 1: n is even $\Rightarrow \cos(2k\pi + x) = \cos(x)$ since $\cos(x)$ repeats on intervals of 2π

Case 2: n is odd $\Rightarrow n + 1$ is even $\Rightarrow \cos(n\pi + x) = \cos((n + 1)\pi + x - \pi) = \cos(x - \pi) = -\cos(x)$

Combining Case 1 and Case 2, we get

$$\cos(n\pi + x) = (-1)^n \cos(x), \forall x \in \mathbb{R} \tag{6}$$

2.3 The Proof

Let $x_0 \in \mathbb{R}$ and let $m \in \mathbb{N}$. If we round $b^m x_0$ to the nearest integer and call it k_m, we get

$$b^m x_0 - \frac{1}{2} \leq k_m \leq b^m x_0 + \frac{1}{2} \tag{7}$$

Now, consider

$$x_m = \frac{k_m + 1}{b^m} \tag{8}$$

Plugging (8) into (7), we get the inequality

$$x_m \geq \frac{b^m x_0 - \frac{1}{2} + 1}{b^m} \geq \frac{b^m x_0}{b_m} = x_0 \tag{9}$$

as well as

$$x_m \leq \frac{b^m x_0 + \frac{1}{2} + 1}{b^m} = x_0 + \frac{3}{2b^m} \tag{10}$$

By combining (9) and (10), we get

$$x_0 \leq x_m \leq x_0 + \frac{3}{2b^m} \tag{11}$$

If we take $\lim_{m\to\infty}$ for each term in the inequality, by the Squeeze Theorem, we get $\lim_{m\to\infty} x_m = x_0$

Now, consider the equation

$$\begin{aligned} W(x_m) - W(x_0) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_m) - \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_0) \\ = \sum_{n=0}^{\infty} a^n (\cos(b^n \pi x_m) - \cos(b^n \pi x_0)) = A + B \end{aligned} \tag{12}$$

where

$$\begin{aligned} A &= \sum_{n=0}^{m-1} a^n (\cos(b^n \pi x_m) - \cos(b^n \pi x_0)) \\ B &= \sum_{n=m}^{\infty} a^n (\cos(b^n \pi x_m) - \cos(b^n \pi x_0)) \end{aligned} \tag{13}$$

Having defined A and B, we now need to find the upper bound for $|A|$ and the lower bound for $|B|$ to proceed with this proof.

2.4 Finding an Upper Bound for $|A|$

By the triangle inequality and (5),

$$\begin{aligned} |A| \leq \sum_{n=0}^{m-1} a^n |\cos(b^n \pi x_m - \cos(b^n \pi x_0)| \\ \leq \sum_{n=0}^{m-1} a^n b^n \pi (x_m - x_0) = \pi (x_m - x_0) \sum_{n=0}^{m-1} (ab)^n \end{aligned} \tag{14}$$

Since $\sum_{n=0}^{m-1}(ab)^n$ is a geometric series, this implies

$$\begin{aligned}|A| &\leq \pi(x_m - x_0)\sum_{n=0}^{m-1}(ab)^n = \pi(x_m - x_0)\frac{1-(ab)^m}{1-ab} \\ &\leq \frac{\pi(ab)^m}{ab-1}(x_m - x_0) \quad \square\end{aligned} \tag{15}$$

2.5 Finding a Lower Bound for $|B|$

By the definition of x_m in (8),

$$\cos(b^n\pi x_m) = \cos(b^n\pi(\frac{k_m+1}{b^m})) = \cos(b^{n-m}(k_m+1)\pi) \tag{16}$$

Since $\cos(x)$ is periodic on intervals of 2π, and since $b^{n-m}(k_m+1)$ is an integer for $n \geq m$ we have by (6),

$$\begin{aligned}&\cos(b^n\pi x_m) = (-1)^{b^{n-m}(k_m+1)} = ((-1)^{b^{n-m}})^{k_m+1} = ((-1)^{b^{n-m}})^{k_m+1} \\ &= (-1)^{k_m+1} = -(-1)^{k_m}\end{aligned} \tag{17}$$

where b^{n-m} is odd, which implies $(-1)^{n-m} = -1$

Additionally, let $z_m = b^m x_0 - k_m$ and assume b^{n-m}. Now consider the equation

$$\begin{aligned}&\cos(b^n\pi x_0) = \cos(b^n\pi(\frac{k_m + b^m x_0 - k_m}{b^m})) \\ &= \cos(b^{n-m}k_m\pi + b^{n-m}z_m\pi)\end{aligned} \tag{18}$$

We now combine equations (17) and (18) into (13) to get

$$\begin{aligned}B &= \sum_{n=m}^{\infty} a^n(-(-1)^{k_m} - (-1)^{k_m}\cos(b^{n-m}z_m\pi)) \\ &= -(-1)^{k_m}\sum_{n=m}^{\infty} a^n(1+\cos(b^{n-m}z_m\pi))\end{aligned} \tag{19}$$

Now, we know that $a^n > 0$ and also that $1+\cos(b^{n-m}z_m\pi) \geq 0$. It follows that all partial terms in (19) are non-negative and therefore that

$$|B| = -(-1)^{k_m}\sum_{n=m}^{\infty} a^n(1+\cos(b^{n-m}z_m\pi)) \geq a^m(1+\cos(z_m\pi)) \tag{20}$$

Since by definition, $z_m = b^m x_0 - k_m$, by (7), $z_m \in [-\frac{1}{2}, \frac{1}{2}] \Rightarrow \pi z_m \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
It follows that $\cos(z_m\pi) \geq 0$ by the properties of cos and that $|B| \geq a^m$

If we rearrange the equation found in (11), we get $\frac{2b^m}{3}(x_m - x_0) \leq 1$
At last, we can combine this equation with $|B| \geq a^m$ to get the following:

$$|B| \geq a^m * 1 \geq a^m * \frac{2b^m}{3}(x_m - x_0) = \frac{2(ab)^m}{3}(x_m - x_0)\square \tag{21}$$

2.6 Assembling the pieces

At last, we are ready to defeat the final boss. We now combine the bounds for $|A|$ and $|B|$ in (15) and (21) as well as the triangle inequality to obtain the following equation:

$$|A+B| \geq |B|-|A| \geq \frac{2(ab)^m}{3}(x_m-x_0)-\frac{\pi(ab)^m}{ab-1}(x_m-x_0) = (ab)^m(\frac{2}{3}-\frac{\pi}{ab-1})(x_m-x_0) \tag{22}$$

By (12),

$$\begin{aligned} &|W(x_m) - W(x_0) = |A+B| \geq (ab)^m(\frac{2}{3} - \frac{\pi}{ab-1})(x_m - x_0) \\ &\Rightarrow |\frac{W(x_m) - W(x_0)}{x_m - x_0}| \geq (ab)^m(\frac{2}{3} - \frac{\pi}{ab-1}) \end{aligned} \tag{23}$$

Almost done! We want the left hand side of (23) to approach ∞ as $m \to \infty$. To do so, we need the conditions $ab > 1$ and $(\frac{2}{3} - \frac{\pi}{ab-1}) \geq 0$. Rearranging our inequality for ab gives us $ab \geq 1 + \frac{3}{2}\pi$, which is part of the hypothesis.

Thus, this means that $\forall x_0 \in \mathbb{R}$

$$\lim_{m\to\infty} |\frac{W(x_m) - W(x_0)}{x_m - x_0}| = +\infty \tag{24}$$

as $x_m \to x_0$ and $m \to \infty$.

Thus, (24) shows that W is not differentiable at any points $x_0 \in \mathbb{R}$,
(ie: It is nowhere differentiable on $\mathbb{R}$) $\square$